3/EH-29 (iii) (Syllabus–2015)

2017

(October)

MATHEMATICS

(Elective/Honours)

(GHS-31)

(**Algebra—II and Calculus—II**)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. (a) Show that the set of nth roots of unity is a group under multiplication of complex numbers. 4

(b) Prove that a non-empty subset H of a group G is a subgroup of G if and only if $a, b \in H$ implies $a b^{-1} \in H$, where b^{-1} is the inverse of b in G. 4

(Turn Over)

(c) Verify whether the binary operation '*' defined on Q by $a * b = \frac{ab}{2}$ is (i) commutative and (ii) associative. 1+2=3

(d) Show that a group G is Abelian if and only if $(ab)^2 = a^2 b^2$ for all $a, b \in G$. 4

2. (a) Prove that every group of prime order is cyclic. Is it Abelian? Justify your answer. 3+1=4

(b) State and prove Lagrange's theorem on the order of a finite group. 1+4=5

(c) Show that the remainder on dividing 7^9 by 15 is 7. State the theorem you have used. 3+1=4

(d) Give an example to show that the union of two subgroups of a group may not be a subgroup. 2

UNIT—II

3. (a) Solve the equation

$$x^4 + x^3 - 16x^2 - 4x + 48 = 0$$

given that the product of two of its roots is 6. 5

(Continued)

(b) Expand $x^5 - 6x^3 + x^2 - 1$ in powers of $x + 1$. 4

(c) If α, β, γ be the roots of the equation $x^3 - ax^2 - bx - c = 0$, find in terms of the coefficients the values of (i) $\Sigma\alpha^2\beta$ and (ii) $\Sigma\alpha^2\beta^2$. 3+3=6

4. (a) Find all the values of $(1+i)^{1/7}$ by De Moivre's theorem. 5

(b) Solve the equation $x^3 - 3x + 1 = 0$ by Cardan method. 6

(c) Find the equation whose roots are the roots of $x^5 + 4x^3 - x^2 + 11 = 0$ each diminished by 3. 4

UNIT—III

5. (a) Prove that if a sequence converges, then its limit is unique. 4

(b) Show that the sequence $\{x_n\}$, where $x_n = \left(1 + \frac{1}{n}\right)^n$ is monotonic increasing. Show also that it is bounded. What can you conclude about the convergence of this sequence? 3+2+1=6

(Turn Over)

(c) Use Cauchy's general principle of convergence to prove that the sequence $\{x_n\}$ converges, when

$$x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$ 5

6. (a) What is an alternating series? Prove that the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges if $\{a_n\}$ is positive monotonic decreasing sequence and $a_n \to 0$ as $n \to \infty$. 2+4=6

(b) Test the convergence of the following series (any *two*) : 3×2=6

(i) $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

(ii) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

(iii) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7 \cdots\cdots (2n-1)}{2\cdot 4\cdot 6 \cdots\cdots 2n(2n+1)}$

(c) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{n+1} x^n$. 3

UNIT—IV

7. (a) State and prove Lagrange's mean value theorem of differential calculus. 1+4=5

(b) Show that $\frac{x}{1+x} < \log(1+x) < x$, for all positive real values of x. 4

(c) Show that $x^{1/x} (x > 0)$ is a maximum at $x = e$ and deduce that $e^{\pi} > \pi^{e}$. 3+1=4

(d) Find the points of inflexion, if any, of the curve $x = (\log y)^3$. 2

8. (a) When is a function $f : D \to \mathbb{R}$ said to be continuous at a point (a, b), where $D \subset \mathbb{R}^2$ and $(a, b) \in D$? Test the continuity of the function defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

at the origin. 1+4=5

(b) Show that

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4 + y^2}$$

does not exist. 2

(Turn Over)

(c) If

$$u = \frac{x^2y^2}{x+y}$$

apply Euler's theorem to find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ and hence deduce that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 6u$$ 2+4=6

(d) State Schwarz's theorem on mixed partial derivative for a real-valued function of two real variables. 2

UNIT—V

9. (a) Expand $f(x) = \sin x$ in a finite series in powers of x with remainder in Cauchy's form. 4

(b) Let $f:[a, b] \to \mathbb{R}$ be a continuous function and $F:[a, b] \to \mathbb{R}$ be a function such that $F'(x) = f(x)$, for all $x \in [a, b]$. Show that $\int_a^b f(x)\,dx = F(b) - F(a)$. 6

(c) Show that the area bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$ is $\frac{16}{3}$ sq unit. 5

10. (a) Evaluate $\iint_C x^2y^2\,dx\,dy$, where

$$C = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$$ 5

(b) Find the length of the arc of the parabola $y^2 = 16x$ measured from the vertex to an extremity of its latus rectum. 5

(c) Find the volume and the surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about x-axis. 5

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